Probability distribution characteristics of chaos in a simple population model and the Bonhoeffer–van der Pol oscillator

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For a chaotic time series $\{x_n\}$ of a simple discrete population model and the Bonhoeffer–van der Pol oscillator, we consider a k-step difference quantity $\Delta x_k = x_{m+k} - x_m$. We study the dynamic behavior of Δx_k. We show that nonstationary probability distribution $P(\Delta x_k)$ occurs for ''weak'' chaos and stationary distribution for ''strong'' chaos. For n-band chaotic attractor switching between n classes of probability distributions is observed. The connection between the degree of nonstationarity and the control parameter, Lyapunov exponent, and correlation dimension is investigated. A power-law dependence is found.
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A dynamical variable of chaotic systems can exhibit a Gaussian as well as a non-Gaussian type of probability distribution [1,2]. In deterministic diffusion the variance $x_{n+1} - x_n$ is of interest. Instead of looking at the probability distribution of x_n and $x_{n+1} - x_n$, in the present paper we consider the probability distributions $P(\Delta x_k)$ $(k=1,2,\ldots)$ of $\Delta x_k = x_{n+k} - x_n$. The connection between $P(\Delta x_k)$ of different k forms a probability association [3]. We investigate the characteristics of $P(\Delta x_k)$ of chaotic attractors in a simple discrete exponential logistic population growth model [4–6]

$$x_{n+1} = f(x_n) = x_n \exp[A(1-x_n)] \tag{1}$$

and the Bonhoeffer–van der Pol (BVP) oscillator [7]

$$\frac{dx}{dt} = x - \frac{x^3}{3} - y + f\cos\Omega t, \quad \frac{dy}{dt} = c(x+a-by), \tag{2}$$

where A, a, b, and c are constant parameters.

The most useful quantity to identify regular and chaotic motions is the spectrum of Lyapunov exponents [1]. For a regular motion, the largest Lyapunov exponent must be negative. At least one positive Lyapunov exponent implies chaos. In the systems (1) and (2) we show that a stationary probability distribution $P(\Delta x_k)$ occurs for strong chaos (characterized by a large positive Lyapunov exponent) and a nonstationary distribution for weak chaos (characterized by a positive but small Lyapunov exponent). This kind of analysis of probability distribution characteristics of chaotic oscillations can be regarded as a simple way of distinguishing different forms of chaos and their geometric structure in ecological data or biological time series where standard dynamical systems theory techniques cannot be applied easily.

Let $\{x_n\}$ $(n=1,2,\ldots,N)$ be the time series of the exponential logistic model (1) or the values of the variable x of the BVP oscillator (2) in the Poincaré map. We consider the k-step difference quantity Δx_k as

$$\Delta x_k = x_{m+k} - x_m, \tag{3}$$

where $m=1,2,\ldots,N', N' \leq N-k$. We calculate the probability distribution of $\{\Delta x_k\}$. Let us consider two such probability distributions $P(\Delta x_k)$ and $P(\Delta x_{k+j})$, which need to be compared. A natural way to compare the two probability distributions is the χ^2 test. The test quantity is defined as

$$\chi^2(k;j) = \sum_{i=1}^{M} \frac{(R_i - S_i)^2}{R_i + S_i}, \tag{4}$$

where R_i and S_i are the probabilities of the ith interval for $P(\Delta x_{k+j})$ and $P(\Delta x_k)$, respectively. In Eq. (4) intervals with $R_i = S_i = 0$ are excluded. If the two probability distributions differ very much we get a large χ^2 value. For two similar distributions the χ^2 value will be small.

First we consider the exponential logistic map (1). Figure 1 shows the variation of the Lyapunov exponent λ as a func-

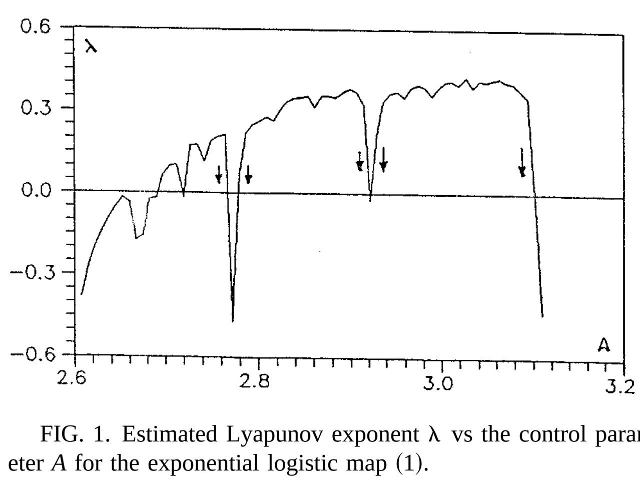

FIG. 1. Estimated Lyapunov exponent λ vs the control parameter A for the exponential logistic map (1).

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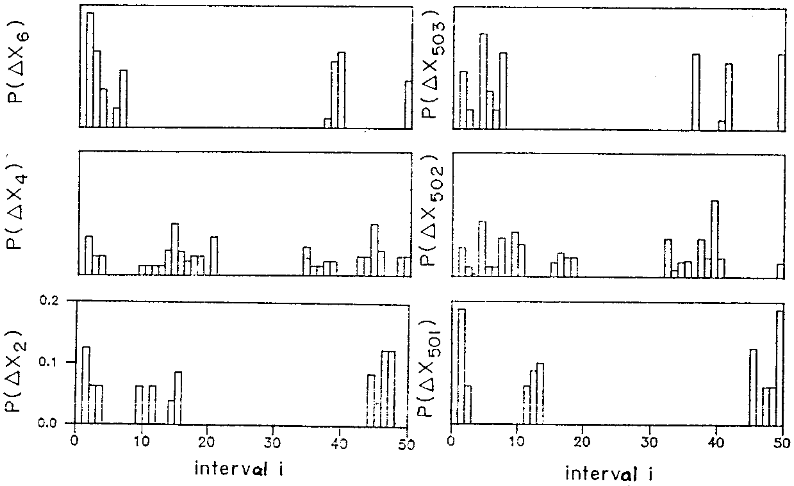

FIG. 2. Numerically calculated $P(\Delta x_k)$ ($k=2,4,6,501,502,503$) for $A=A_{crit}\approx 2.692\,368\,900\,3\ldots$. P is nonstationary.

tion of A. The one-dimensional Lyapunov exponent λ of the attractors of Eq. (1) is estimated using the relation

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|(1-Ax_n)\exp[A(1-x_n)]|. \tag{5}$$

The value of λ is negative for regular motion and it is positive in the chaotic regions. When the parameter A is increased from a small value, a period-doubling route to chaos is found [4]. The onset of chaos is observed at $A_{crit}\approx 2.692\,369\,300\,3\ldots$. In Fig. 1 there are many regions labeled by an arrow, where the Lyapunov exponent is positive but very small, which indicates that the system (1) exhibits weak chaos at these regions. The values of P are calculated for various values of λ.

In the numerical simulation we neglected first 5000 iterations as transient and used the next 10^4 iterations for analysis. Figure 2 shows the evolution of probability distributions P for $k=2$, 4, 6, 501, 502, and 503 of a critical 2^∞ attractor at $A=A_{crit}$. From Fig. 2 it is evident that P changes with k. That is, the distribution is nonstationary. This is further verified by numerically calculating χ^2 using Eq. (4) and is plotted in Fig. 3. The analysis is carried for k values up to 5000. The nondecreasing χ^2 implies the nonstationary characteristics of P. In Eq. (1) nonstationary P is found for weak chaos. As an example, for $A=3.102\,439$ a chaotic attractor with $\lambda=0.0082$ is found and the calculated χ^2 quantity is nondecaying, implying a nonstationary probability distribution. Thus a complete probability distribution is impossible for weak chaos.

A possible mechanism of nonstationary probability distribution can be a recurrence of memory loss and recovery of initial conditions [8,9]. The key to searching for memory recovery is the value of the Lyapunov exponent. Chern and Otsuka [10] applied information theory and a local Lyapunov exponent to characterize the locally deforming nature in chaos. Particularly, using self-information flow and mutual information flow they have shown that memory recovery is possible for chaos with a very small positive Lyapunov exponent. Thus the physical mechanism of the nonstationary probability distribution is the recurrence of memory loss and the recovery of initial conditions [8,9].

A stationary probability distribution is found for strong chaos. For $A=2.838$ the Lyapunov exponent of the chaotic attractor is 0.347. The calculated χ^2 is plotted in Fig. 4. After $k=120$ there is almost no change in χ^2. In other words, the motion is strongly chaotic and the distribution has evolved into a stationary state. This suggests that the variable x_n can be described as if it were generated by a random number generator with a certain probability distribution. A possible mechanism for stationary probability distribution is a complete loss of memory of initial conditions.

As another example for stationary P we consider a two-band chaotic attractor. By two-band chaos we mean that in the x_n versus x_{n+1} plane after transient evolution the iterated values fall on two separate regions. In the exponential logistic map (1) a two-band chaotic attractor is found for $A=2.832$. Figure 5 shows P for $k=31$, 32, 33, and 34. A simple switching between two classes of probability distributions can be seen. $\chi^2(k,1)$ and $\chi^2(k,2)$ are also calculated.

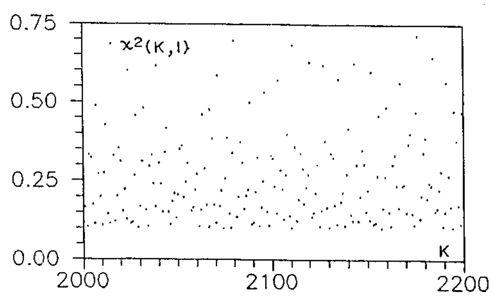

FIG. 3. $\chi^2(k,1)$ vs k for $A=A_{crit}$.

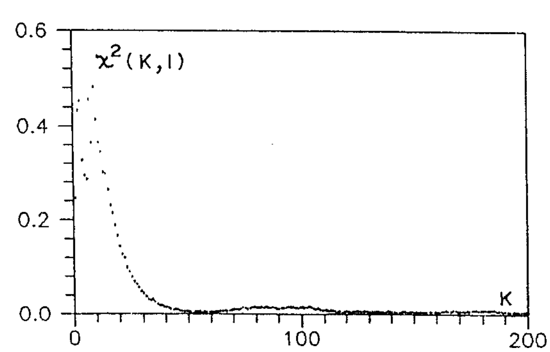

FIG. 4. $\chi^2(k,1)$ vs k for $A=2.838$.

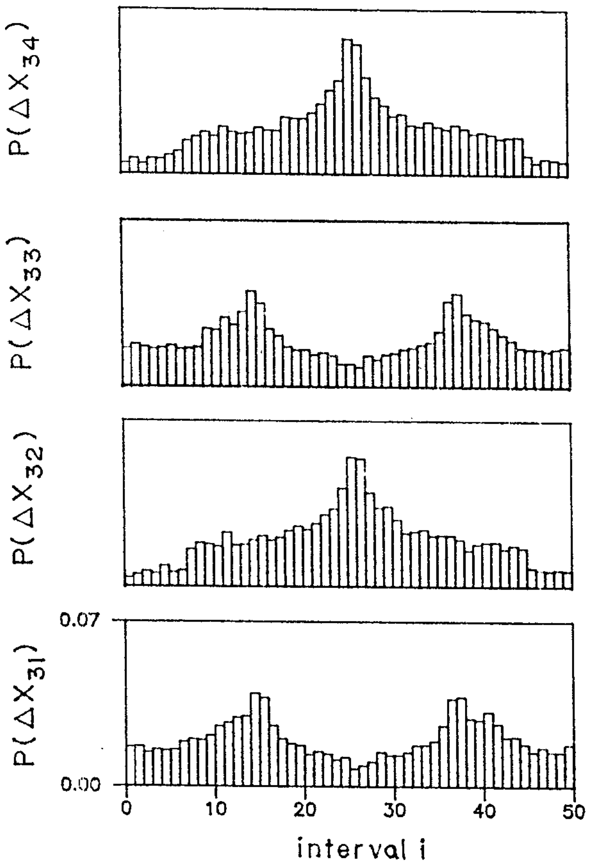

FIG. 5. $P(\Delta x_k)$ ($k=31,32,33,34$) for $A=2.832$ (two-band chaos).

$\chi^2(k,1)$ values are found to be large, while $\chi^2(k,2)$ values become almost zero for large values of k, confirming the stationary characteristic of two different distribution patterns.

Further, nonstationary P occurs for a range of values of the control parameter for which the Lyapunov exponent is relatively small. We have investigated the dependence of the degree of nonstationarity of P with the control parameter, Lyapunov exponent, and correlation dimension. For this purpose, we define the quantity S, the degree of nonstationarity of $P(\Delta x_k)$, as

$$S=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\chi^2(k;j), \tag{6}$$

where $\chi^2(k;j)$ is given by Eq. (4). For strong chaos, $\chi^2\to 0$ for large k and hence $S=0$. Since χ^2 does not decay to zero for weak chaos in the limit $k\to\infty$, evidently S is nonzero. The magnitude of S characterizes the degree of different nonstationary probability distributions. S is calculated for a range of values of control parameters in the exponential logistic and tent maps.

First, we present our result on the tent map for which the one-dimensional Lyapunov exponent λ can be calculated analytically. The tent map is given by

$$x_{n+1}=2\mu\times\begin{cases} x_n, & 0.0\leqslant x<0.5\\ 1-x_n, & 0.5\leqslant x\leqslant 1.0,\end{cases} \tag{7}$$

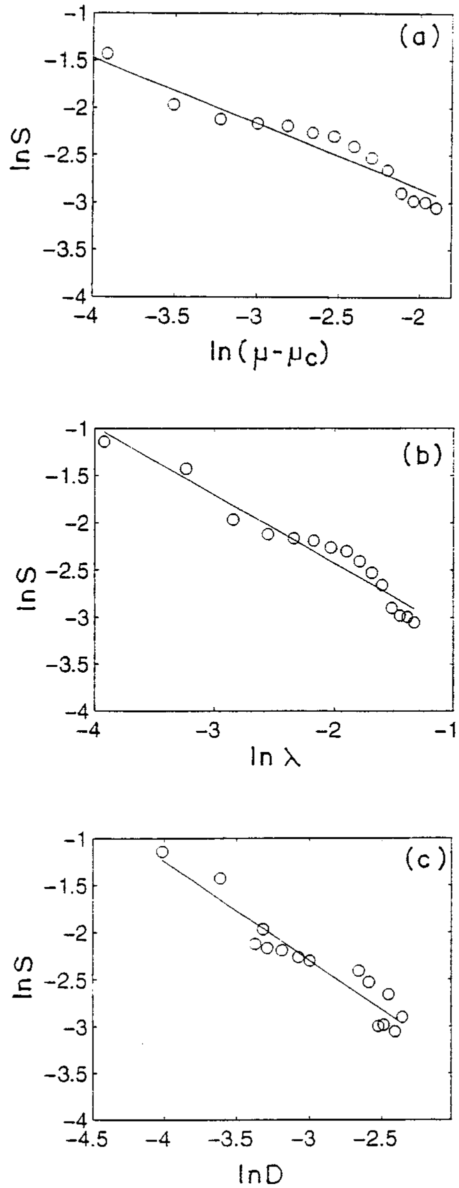

FIG. 6. Variation of S as a function of (a) $\mu-\mu_c$, (b) the Lyapunov exponent λ, and (c) the correlation dimension D in the ln-ln scale for the tent map. Circles represent numerical data and the continuous line is the best straight line fit.

where μ is a control parameter. For $\mu\leqslant 0.5$, $x^*=0$ is a stable fixed point. Chaotic motion occurs for $\mu>0.5$. From the tent map (7) its Lyapunov exponent is $\lambda=\ln 2\mu$.

Thus λ of the attractors of the tent map can be calculated without iterating the map equation (7). At $\mu=\mu_c=0.5$, λ is zero, while for $\mu>0.5$, λ is positive. The nonstationary probability distribution P is found for μ near 0.5, while stationary P is observed for sufficiently large values of μ. Figure 6(a) shows S versus $\mu-\mu_c$ in the ln-ln scale. A power-law variation of S with $\mu-\mu_c$ is observed. S is found to approach zero with $(\mu-\mu_c)^{\alpha}$, where α is a constant. We found $S\approx 0.0144(\mu-\mu_c)^{-0.69}$. Figure 6(b) depicts the variation of S with λ. We note that as λ increases S decays to zero. That is, a transition from nonstationary P to stationary P occurs as λ increases from zero. S is found to scale with λ as $S\approx 0.0206\lambda^{-0.73}$. Further, the correlation dimension D of

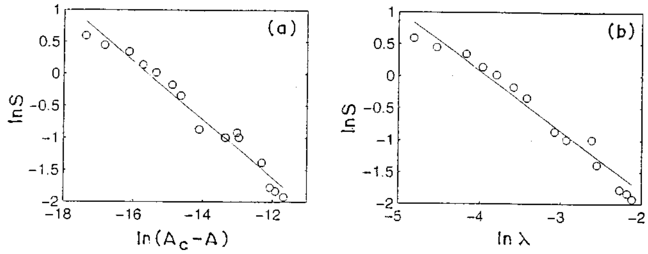

FIG. 7. (a) S vs $A_c - A$ and (b) S vs λ in the ln-ln scale for the exponential logistic map where $A_c = 2.916\,100\,48\ldots$. We found $S = 0.838(A_c - A)^{-0.455} \times 10^{-5}$ and $S = 0.0255\lambda^{-0.936}$.

the attractor of the tent map is calculated by employing the Grassberger-Procaccia [11] algorithm. Figure 6(c) shows the $\ln S$ versus $\ln D$ plot. Here again the quantity S is found to exhibit a power-law dependence on D ($S \approx 0.0043D^{-1.048}$).

We have calculated the quantity S for the exponential logistic map (1). For A values slightly above $A_c \approx 2.916\,100\,448$ periodic motion occurs with λ being negative. At $A = A_c$, $\lambda \approx 0$. Nonstationary P is observed for A values just below A_c. Figures 7(a) and 7(b) depict the variation of S as a function of $A_c - A$ and λ in the ln-ln scale. A power-law dependence of S on $A_c - A$ and λ is clearly seen.

Finally, we have also investigated the probability distribution in the BVP oscillator (2). Here the x values collected at $t = 2\pi n/\Omega$ $(n = 1,2,\ldots)$ are used for analysis. We present part of the results in terms of χ^2. Figure 8 shows χ^2 as a function of k for $f = 1.0919$, $a = 0.7$, $b = 0.8$, $c = 0.1$, and $\Omega = 1.0$, where the system (2) exhibits weak chaos. The corresponding Lyapunov exponent value is approximately equal to 0.0004. The nondecreasing χ^2 clearly indicates the nonstationary probability distributions. We have numerically calculated χ^2 for one-, two-, and four-band chaotic attractors. For two-band and four-band chaotic attractors $\chi^2(k,1)$ values are large (not shown here). However, for two-band and four-band attractors $\chi^2(k,2)$ and $\chi^2(k,4)$, respectively, decay to zero as k increases. For a two- (four-) band attractor simple switching between two (four) different classes of P is found.

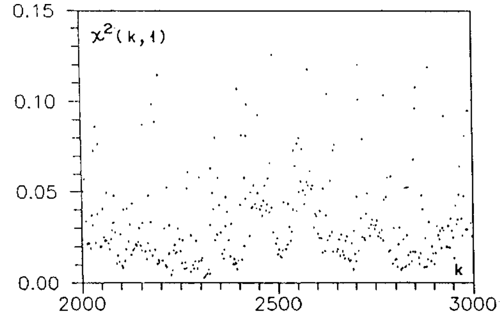

FIG. 8. $\chi^2(k,1)$ vs k for the BVP oscillator (2) with $f = 1.0919$ (weak chaos).

In the present work we have studied the characteristics of the probability distribution P of chaotic attractors of the exponential logistic map (1) and the BVP oscillator (2), which are of ecologically and biologically important systems, respectively. A stationary probability distribution is found for chaotic attractors with a large positive Lyapunov exponent, which corresponds to strong chaos. A nonstationary probability distribution is found to occur for chaotic attractors with sufficiently small positive Lyapunov exponent, which is related to weak chaos. We have also studied the dependence of the degree of nonstationarity S on the Lyapunov exponent, correlation dimension, and control parameter. S is found to approach zero as these quantities increase. A power-law variation of S is found. The above study suggests that the analysis of probability distribution characteristics of chaos can be used to distinguish weak and strong chaos exhibited by the systems under consideration.

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